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ON DIFFERENTIAL GAMES  
WITH INTEGRAL PAYOFF

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SUMMARY

↓  
Using methods of the calculus of variations, ~~the authors~~ *were employed to* obtain necessary conditions that differential games of a certain type have a saddle-point. By strengthening the necessary conditions, sufficient conditions for the existence of a saddle-point and a method of constructing the saddle-point ~~are~~ *derived.* ( ) ↑

# ON DIFFERENTIAL GAMES WITH INTEGRAL PAYOFF

L. D. Berkovitz  
W. H. Fleming

## 1. INTRODUCTION

This paper treats a certain class of two-person zero-sum games which may be described intuitively as follows. Each player chooses a set of instructions for play which are called strategies and which tell him how to choose a number from the unit interval at each instant of time  $t$  lying in an interval  $0 \leq t \leq T$ . If  $y(t)$  denotes the number chosen by Player I at time  $t$  and  $z(t)$  the number chosen by Player II, then the position or state of the game is determined by the differential equation

$$\dot{x} = \frac{dx}{dt} = g(x, y, z)$$

together with the initial position  $x(0) = c$ . The strategies or instructions for play may take into account the state  $x(t)$  of the game at time  $t$ , it being assumed that each player always knows the state of the game. The payoff to Player I is given by the integral

$$P = \int_0^T f[x(t), y(t), z(t)] dt.$$

Player I's objective is to choose a strategy which enables him to maximize  $P$ , while the second player's goal is a strategy which will minimize  $P$ .

To give a precise, useful mathematical formulation of the problem is a nontrivial task, and the present paper by no means

attempts to give a complete theory. There is not even an existence theorem ensuring that  $\max \min = \min \max$ . Some of the results which we shall present were first obtained by Isaacs [6], [7], in a formal manner or else under more restrictive conditions than those set forth here. The reader will also note a connection with the work of Bellman [1]. What we do is to consider the problem as a two-sided extremum problem with differential and inequality side conditions and apply well-known techniques of the calculus of variations (c.f. Bliss [3], Valentine [8]) to see how much information can be obtained in this way. Analogues of the classical Euler equations are derived as necessary conditions for a saddle-point. Their solutions, which are called extremals, turn out to be characteristics of a certain partial differential equation involving the value of the game. The converse problem is then treated. Namely, when does a family of extremals determine a solution of the game? The basic condition for this is that the curves  $x(t)$  associated with the family  $P$  of extremals simply cover a certain region  $R$  in  $(x,t)$  plane. The family then defines a pair of strategies which are optimal in a sense to be made precise below. The value  $W$  of the game is a continuously differentiable function of position  $(c,t)$  in  $R$ . Actually, we consider the more general case of a finite number of nonoverlapping regions  $R_1, \dots, R_m$ , such that, for each  $i$ ,  $R_i$  is simply covered by a family  $P_i$  of extremals. In this case the value  $W$  is continuously differentiable in each region  $R_i$  and is continuous across boundary arcs common to more than one region. The partial derivatives of  $W$  are in general

discontinuous across such boundary arcs.

Although the discussion in this paper is carried out for a scalar differential equation,  $\dot{x} = g$ , the argument can be carried over to a vector equation  $\dot{x} = 0$ .

## 2. MATHEMATICAL FORMULATION

We begin by defining certain terms which are used throughout this paper. An arc in the  $(x,t)$  plane is called smooth if each function in its representation in terms of arc length is twice continuously differentiable. A region  $G$  in the  $(x,t)$  plane has a piecewise smooth boundary if its boundary consists of a finite number of smooth arcs without cusps at corners. Let  $\bar{G}$  denote the closure of  $G$ . A function  $\phi(x,t)$  is  $C^{(k)}$  in  $\bar{G}$  if it is  $C^{(k)}$  in  $G$  and all of its derivatives up to and including those of order  $k$  have a continuous extension to  $\bar{G}$ .

Throughout this paper we shall be considering two real-valued functions  $f(x,y,z)$  and  $g(x,y,z)$  which are defined for all  $x$  and for all  $y,z$  with  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$ , and which satisfy the following conditions:

- (a)  $f$  and  $g$  are  $C^{(2)}$  on the closure of their domain of definition;
- (2.1) (b)  $f_{yy} \leq 0$ ,  $f_{zz} \geq 0$ ,  
 $g_{yy} = 0$ ,  $g_{zz} = 0$ ;
- (c)  $ax + b \leq g(x,y,z) \leq a'x + b'$  for suitable constants  $a, a', b, b'$ .

Let  $T > 0$  be fixed for the remainder of this paper. The symbol  $R$  will designate a region of the  $(x,t)$  plane contained in the strip  $0 < t < T$ , such that the projection of  $R$  on the  $t$  axis covers the entire interval  $0 < t < T$ . We shall consider functions  $y(x,t)$  and  $z(x,t)$  defined on  $\bar{R}$  and the differential equation

$$(2.2) \quad \dot{x} = \frac{dx}{dt} = g[x, y(x,t), z(x,t)]$$

with initial condition

$$(2.3) \quad x(t_0) = c, \quad 0 \leq t_0 < T \text{ and } (c, t_0) \text{ in } \bar{R},$$

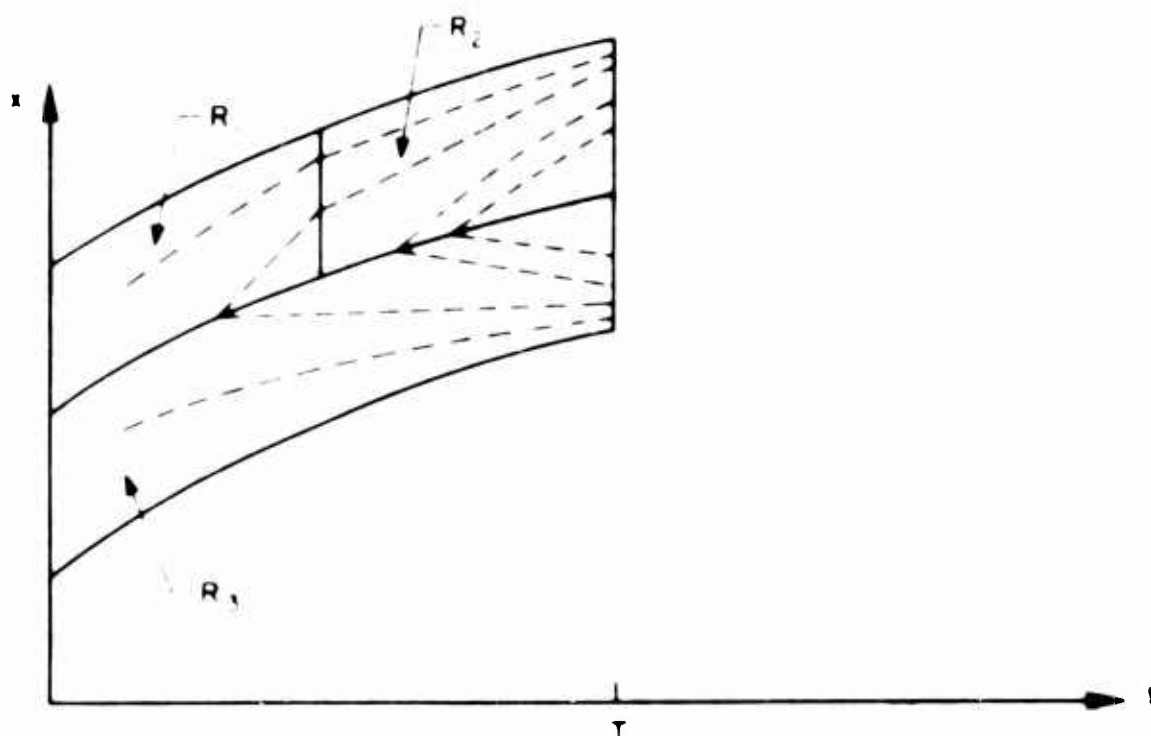
as follows. Let  $\mathcal{Y}$  denote a class of functions  $y(x, t)$ , and  $\mathcal{Z}$  a class of functions  $z(x, t)$ , defined in  $\bar{R}$ , such that for each  $y$  in  $\mathcal{Y}$  and  $z$  in  $\mathcal{Z}$  the following properties hold:

- (a)  $0 \leq y(x, t) \leq 1, \quad 0 \leq z(x, t) \leq 1.$
- (b) There exists a decomposition (which may depend on the pair  $(y, z)$ ) of the region  $\bar{R}$  into non-overlapping<sup>1</sup> subregions  $\bar{R}_1, \dots, \bar{R}_n$  with piecewise smooth boundaries such that, in each open region  $R_1$ ,  $y$  and  $z$  coincide with functions that are  $C^{(1)}$  in  $\bar{R}_1$ .
- (c) For each  $(c, t_0)$  belonging to  $\bar{R}$ , the equation (2.2) subject to (2.3) has a finite number  $\alpha \geq 1$  of solutions  $x_1(t), \dots, x_\alpha(t)$ , each satisfying a Lipschitz condition and lying in  $\bar{R}$  for  $t_0 \leq t \leq T$ . Furthermore, with the possible exception of finitely many values of  $t$ , the point  $(x_j(t), t)$ ,  $j = 1, \dots, \alpha$ , belongs to just one region  $\bar{R}_1$ .
- (d) If  $(c, t_0)$  belongs to just one region  $\bar{R}_1$ , then  $\alpha = 1$ .

The accompanying figure may help indicate what we have in mind. In this case  $\bar{R} = \bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_3$ . The dotted lines indicate the curves  $x(t)$  resulting from (2.2).

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<sup>1</sup> By nonoverlapping subregions we mean, as usual, subregions such that no two have a common interior point.



Except for  $(c, t_0)$  on the boundary separating  $R_1$  and  $R_2$ , the path is unique. On the boundary there are two curves, one entering  $R_1$  or  $R_2$  and the other entering  $R_3$ .

At this point it should be noted that if  $(c, t_0)$  is in an open region  $R_1$ , then the assumptions that  $g$  is in  $C^{(2)}$  and that  $y$  and  $z$  are in  $C^{(1)}$  on  $R_1$ , coupled with the standard existence and uniqueness theorem for ordinary differential equations, guarantee that there is a unique solution through  $(c, t_0)$ . From the fact that  $\alpha$  is not less than 1 and is finite, and from the uniqueness of the solution of (2.2) - (2.3) in  $R_1$ , it follows that on each  $R_1$  the solution of (2.2) - (2.3) is unique. This uniqueness and the assumption that each solution meets the sets  $R_1 \cap R_j$  in at most finitely many points have as a consequence the fact that at each point of  $R$  the index  $d$  cannot exceed the number of regions  $R_1$  meeting at that point.



We shall sometimes call the elements  $y, z$  of  $\mathcal{Y}, \mathcal{Z}$  strategies. There is a certain element of artificiality in this definition in that a strategy  $y$  in  $\mathcal{Y}$  may no longer be a strategy if  $\mathcal{Z}$  is replaced by a different class  $\mathcal{Z}'$  of functions  $z(c, t)$ . In order to get a satisfactory general notion of strategy for games of this type, it may be necessary to resort to notions similar to that of  $K$ -strategy, in the terminology of Isaacs [6]. It is proved in [6] and also in [5] that if the value  $W$  of the game, a notion to be defined precisely below, is a continuously differentiable function of position  $(x, t)$ , then  $\epsilon$ -effective  $K$ -strategies exist. It would be interesting to extend this result to the present case in which  $W$  will turn out to be only piecewise continuously differentiable.

Let  $y \in \mathcal{Y}$ ,  $z \in \mathcal{Z}$ , and let  $(c, t_0)$  be a point of  $\bar{R}$  belonging to just one subregion  $R_1$ . Let

$$(2.5) \quad y(t) = y[x(t), t], \quad z(t) = z[x(t), t],$$

where  $x(t)$  is the solution of (2.2) - (2.3). The functions  $x(t)$ ,  $y(t)$ ,  $z(t)$  determined by (2.2), (2.3) and (2.5) will be called the path corresponding to the pair of strategies  $(y, z)$ . Define the payoff as follows:

$$(2.6) \quad P(y, z; c, t_0) = \int_{t_0}^T f[x(t), y(t), z(t)] dt.$$

$P$  is to be thought of as the payoff to Player I, and  $-P$  the payoff to Player II. If  $(c, t_0)$  belongs to more than one region  $R_1$ , there are, in general, finitely many solutions  $x_1(t), \dots, x_K(t)$  of (2.2) through  $(c, t_0)$ . Let  $P_{(\lambda)}(y, z; c, t_0)$  denote the payoff when the solution  $x_K(t)$ ,  $K = 1, \dots, K$ , is used.

A pair of functions  $(y^*, z^*)$  with  $y^* \in \mathcal{Y}$  and  $z^* \in \mathcal{Z}$  is said to be a saddle-point relative to  $(\mathcal{Y}, \mathcal{Z})$  and  $(c, t_0)$  if:

- (a)  $P_{(\kappa)}(y^*, z^*; c, t_0)$  is the same for all  $\kappa = 1, \dots, \alpha$ .  
Call this value  $W(c, t_0)$ .
- (2.7) (b) For all  $y \in \mathcal{Y}$ ,  $z \in \mathcal{Z}$ , all indices  $\mu$  which may arise from the pairs  $(y, z^*)$  at  $(c, t_0)$ , and all indices  $\nu$  which may arise from the pairs  $(y^*, z)$  at  $(c, t_0)$ , we have

$$P_{(\mu)}(y, z^*; c, t_0) \leq W(c, t_0) \leq P_{(\nu)}(y^*, z; c, t_0).$$

Henceforth a path corresponding to a saddle-point  $(y^*(x, t), z^*(x, t))$  will be denoted by  $\bar{x}(t)$ ,  $\bar{y}(t)$ ,  $\bar{z}(t)$ . Also, whenever  $\alpha = 1$  we shall drop the subscript on the payoff  $P$ .

We close this section with some remarks about the assumptions on  $f$  and  $g$ . Conditions 2.1(b) say that  $f$  is concave in  $y$  and convex in  $z$ , and are imposed in order to avoid introducing mixed strategies. In similar problems involving only maximization or only minimization, such concavity or convexity requirements were found to be essential to ensure that the maximum or minimum is attained. (See [2].) The requirement (2.1)(c) is placed on  $g$  to ensure that the solutions of (2.2) and (2.3) have a uniform bound depending only on the initial condition  $(c, t_0)$ .

### 3. NECESSARY CONDITIONS

We now proceed to derive a set of necessary conditions which must hold along a path resulting from a saddle-point. In simple examples, use of these necessary conditions, together with the sufficiency theory to follow in Sections 4 and 5, often enables one to find the complete solution to the game. (See Section 7.)

Let  $(y^*(x,t), z^*(x,t))$  be a saddle-point relative to  $(\mathcal{U}, \mathcal{V})$  and  $(c,0)$ , where  $(c,0)$  belongs to just one region  $R_1$ , and let it be assumed that:

For  $0 < t < T$  the curve  $\bar{x}(t)$  lies in the open region  $R$ , the regions  $R_1, \dots, R_n$  are numbered so that the curve  $\bar{x}(t)$  has a nonnull intersection with the first  $k+1 \leq n$  regions  $R_1$ , and  $R_1 \wedge R_{1+1}$  meets  $\bar{x}(t)$  for  $t = t_1, 0 < t_1 < \dots < t_k < T$ . Furthermore, each sufficiently small neighborhood of  $(\bar{x}(t_1), t_1)$  intersects  $R_1 \wedge R_{1+1}$  in a smooth arc  $\gamma_1(t)$  which is not tangent to  $\bar{x}(t)$  at  $t = t_1$ .

Since  $g$  is of class  $C^{(2)}$  and  $y^*$  and  $z^*$  are  $C^{(1)}$  on each  $R_1$ , it follows from the standard existence and uniqueness theorem for ordinary differential equations that the equation

$$\dot{x} = g[x, y^*(x,t), z^*(x,t)] ,$$

subject to the initial condition  $x(t_0) = x_0$ , has a unique solution  $\bar{x}(t; x_0, t_0)$  through each point  $(x_0, t_0)$  of  $R_1, i = 1, \dots, n$ . The nontangency condition in (3.1) ensures that for  $(x'_0, t'_0)$  sufficiently close to  $(x_0, t_0)$  the solution  $\bar{x}(t; x'_0, t'_0)$  has a unique continuation

across each arc  $\gamma_i(t)$ ,  $i = 1, \dots, k$ . This fact together with (2.1)(c) implies the existence of a constant  $K$  such that for  $(x'_0, t_0)$  sufficiently close to  $(x_0, t_0)$  the inequality

$$|\bar{x}(t; x'_0, t_0) - \bar{x}(t; x_0, t_0)| \leq K|x_0 - x'_0|$$

holds for  $t_0 \leq t \leq T$ .

The case in which  $\bar{x}(t)$  follows some boundary  $\bar{R}_1 \wedge \bar{R}_{1+1}$  over an interval of time rather than crossing as in (3.1) is also of importance, but is not treated here.

Let us introduce the following quantities, which play a fundamental role in what follows:

$$\begin{aligned} H(t) &= f_y[\bar{x}(t), \bar{y}(t), \bar{z}(t)] + \lambda(t)g_y[\bar{x}(t), \bar{y}(t), \bar{z}(t)], \\ K(t) &= f_z[\bar{x}(t), \bar{y}(t), \bar{z}(t)] + \lambda(t)g_z[\bar{x}(t), \bar{y}(t), \bar{z}(t)], \end{aligned} \quad (3.2)$$

where  $\lambda(t)$  is a multiplier determined by

$$\begin{aligned} \dot{\lambda}(t) &= -f_x[\bar{x}(t), \bar{y}(t), \bar{z}(t)] - \lambda g_x[\bar{x}(t), \bar{y}(t), \bar{z}(t)], \\ \lambda(T) &= 0. \end{aligned} \quad (3.3)$$

Theorem 1. Let  $(y^*(x, t), z^*(x, t))$  be a saddle-point relative to  $y, z$  for the initial condition  $(c, 0)$ , and let (3.1) hold. Then for  $t \neq t_1$ ,  $i = 1, 2, \dots, k$ , the following necessary condition is satisfied:

$$\begin{aligned} H(t) &\geq 0 \text{ whenever } \bar{y}(t) = 1, & K(t) &\leq 0 \text{ whenever } \bar{z}(t) = 1 \\ (3.4) \quad &= 0 \text{ whenever } 0 < \bar{y}(t) < 1, & &= 0 \text{ whenever } 0 < \bar{z}(t) < 1, \\ &\leq 0 \text{ whenever } \bar{y}(t) = 0; & &\leq 0 \text{ whenever } \bar{z}(t) = 0. \end{aligned}$$

We note that Theorem 1 is the exact analogue of the corresponding necessary condition for maximum problems [2, formula (4.2)]. In the latter case, however, it is not necessary to make any restrictive assumptions such as (3.1).

The proof of Theorem 1 is begun by following the common variational technique of finding

$$(3.5) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} [P(y^* + \varepsilon \eta, z^*) - P(y^*, z^*)] ,$$

where  $\varepsilon \eta(x, t)$  is an "admissible variation" for small  $\varepsilon > 0$ . Specifically, let  $\eta(x, t)$  be  $C^{(1)}$  in  $R$ , vanish identically outside of one of the subregions  $R_1$ , and satisfy  $0 \leq y^* + \eta \leq 1$ . Let  $x_\varepsilon$  denote the solution of

$$\dot{x} = g[x, y^* + \varepsilon \eta, z^*] , \quad x(0) = c .$$

From the assumptions concerning  $\eta(x, t)$  and the discussion immediately after (3.1) it is not difficult to see that there is a constant  $K$  such that for  $0 \leq t \leq T$  we have

$$|x_\varepsilon(t) - \bar{x}(t)| \leq K\varepsilon .$$

For brevity let us write

$$A = f_x + f_y y_x^* + f_z z_x^* ,$$

$$B = g_x + g_y y_x^* + g_z z_x^* .$$

Define  $\xi(t)$  by the equations

$$\dot{\xi} = B\xi + g_y \eta , \quad \xi(0) = 0 .$$

By standard arguments [2] the limit (3.5) exists and equals

$$\int_0^T (A\xi + f_y \eta) dt.$$

Define  $\lambda_1(t)$  by

$$\dot{\lambda}_1(t) = -(A + B\lambda_1), \quad \lambda_1(T) = 0.$$

Integration by parts gives [2, Section 4]

$$\int_0^T (A\xi + f_y \eta) dt = \int_0^T (f_y + \lambda_1 g_y) \eta dt.$$

Since  $(y^*, z^*)$  is a saddle-point,

$$P(y^* + \epsilon \eta, z^*) - P(y^*, z^*) \leq 0$$

for sufficiently small  $\epsilon$ . Hence

$$\int_0^T (f_y + \lambda_1 g_y) \eta dt \leq 0$$

for all  $\eta$ . It follows that since  $\eta$  is arbitrary,

$$(3.6) \quad f_y + \lambda_1 g_y \begin{cases} \geq 0 & \text{whenever } \bar{y} = 1, \\ = 0 & \text{whenever } 0 < \bar{y} < 1, \\ \leq 0 & \text{whenever } \bar{y} = 0, \end{cases}$$

provided  $t \neq t_1$ . A similar argument shows that

$$(3.7) \quad f_z + \lambda_1 g_z \begin{cases} \leq 0 & \text{whenever } \bar{z} = 1, \\ = 0 & \text{whenever } 0 < \bar{z} < 1, \\ \geq 0 & \text{whenever } \bar{z} = 0, \end{cases}$$

provided  $t \neq t_1$ . However,

$$(3.8) \quad A + B\lambda_1 = f_x + g_x \lambda_1 + y^*(f_y + \lambda_1 g_y) + z^*(f_z + \lambda_1 g_z).$$

From (3.6), (3.7), and the fact that for  $t \neq t_1$   $y_x^*$  and  $z_x^*$  vanish whenever  $y^* = 0, 1$  and  $z^* = 0, 1$ , respectively, it follows that along the path  $\bar{x}(t)$ ,  $\bar{y}(t)$ ,  $\bar{z}(t)$ , the third and fourth terms of (3.8) vanish. Hence

$$\dot{\lambda}_1 = -(A + B\lambda_1) = -(f_x + \lambda_1 g_x), \quad \lambda_1(T) = 0.$$

Since this is just the defining equation for  $\lambda(t)$ , it follows that  $\lambda_1(t) = \lambda(t)$ , and the theorem is proved.

#### 4. AN INVARIANT INTEGRAL

This section and the two succeeding sections will be devoted to developing sufficient conditions for the existence of a saddle-point. Let us call any solution  $\bar{x}(t), \bar{y}(t), \bar{z}(t)$  of the necessary conditions of Theorem 1, where  $\bar{x}, \bar{y}, \bar{z}$  are further related by  $\dot{\bar{x}} = g(\bar{x}, \bar{y}, \bar{z})$ ,  $\bar{x}(0) = c$ , and extremal through the initial condition  $(c, 0)$ . For a maximization problem (i.e.,  $z$  absent) the solution may be given in terms of functions of time only and is an extremal, although not every extremal need maximize (c.f. [2], [4]). For the game, however, examples show (see Section 7) that the solution cannot be found, in general, in terms of functions of time only. Hence to construct saddle-points it is at once necessary to consider families of extremals through a variety of initial conditions. The problem before us is to determine when a one-parameter family of extremals actually yields a saddle-point.

We begin by defining an analogue of the Hilbert invariant integral of the calculus of variations. In addition to the conditions imposed on  $R$  in Section 2, let  $R$  be simply connected and have a piecewise smooth boundary, and let  $y^*(x, t), z^*(x, t), \Lambda(x, t)$  be three functions which satisfy the following conditions:

- (a)  $0 \leq y^*(x, t) \leq 1, \quad 0 \leq z^*(x, t) \leq 1;$
- (b)  $y^*, z^*, \Lambda$  are  $C^{(1)}$  in  $\bar{R}$ ;
- (4.1) (c) for all  $(c, t)$  in  $\bar{R}$  the game over the square  
 $0 \leq y \leq 1, 0 \leq z \leq 1$ , with payoff

$$\Phi(x, t; y, z) = f(x, y, z) + \Lambda(x, t)g(x, y, z) .$$

has  $(y^*, z^*)$  as saddle-point; and



$$(d) \quad \Lambda_x(x,t)g(x,y^*,z^*) + \Lambda_t(x,t) \\ = -[f_x(x,y^*,z^*) + \Lambda(x,t)g_x(x,y^*,z^*)] .$$

An immediate consequence of (4.1)(c) is the following:

$$(4.2) \quad y_x^*(f_y + \Lambda g_y) - z_x^*(f_z + \Lambda g_z) = 0,$$

where the derivatives of  $f$  and  $g$  are evaluated at  $(x,y^*,z^*)$ .

Let

$$\Psi(x,t) = f(x,y^*,z^*) + \Lambda(x,t)g(x,y^*,z^*).$$

Then

$$\Psi_x = [f_x + g_x] + y_x^*[f_y + \Lambda g_y] + z_x^*[f_z + \Lambda g_z] + \Lambda_x g.$$

From (4.2) and (4.1)(d) we get

$$(4.3) \quad \Psi_x = f_x + g_x + \Lambda_x g = -\Lambda_t .$$

Define

$$(4.4) \quad W(x,t) = \int_{(x,t)} \Psi dt - \Lambda dx, \quad (x,t) \text{ in } \bar{R} ,$$

where the upper limit is a fixed but arbitrary point in  $\bar{R}$  and the integral is taken over an arbitrary path which, perhaps excluding end points, lies in  $R$ . In view of (4.3) it is clear that (4.4) is independent of path, and so  $W(x,t)$  is well defined in  $R$ . Clearly,

$$(4.5) \quad W_t = -\Psi, \quad W_x = \Lambda.$$

The following lemma is an immediate consequence of the preceding discussion.

Lemma 1. For all  $(x,t)$  in  $\bar{R}$  and  $0 \leq y \leq 1$ ,  $0 \leq z \leq 1$ , we have

$$(4.6) \quad f(x,y,z^*) + W_x g(x,y,z^*) \leq -W_t \leq f(x,y^*,z) + W_x g(x,y^*,z) .$$

## 5. FIELDS

The notion of a field, which is taken from the calculus of variations, will now be introduced. Let  $R$  be a region with piecewise smooth boundary contained in the strip  $0 < t < T$  of the  $(x,t)$  plane as described in Section 2, and such that:

- (5.1) (a) There is a decomposition of  $R$  into a finite number of nonoverlapping regions  $R_j$ , such that  $R = R_1 \cup R_2 \cup \dots \cup R_n$ , and such that each  $R_j$ ,  $j = 1, \dots, n$ , is simply connected, has piecewise smooth boundary, and has functions  $y_j(x,t)$ ,  $z_j(x,t)$ ,  $\lambda_j(x,t)$  defined on it which satisfy conditions (4.1).
- (b) There are functions  $y^*(x,t)$ ,  $z^*(x,t)$  which satisfy (2.4)(a), (2.4)(c), (2.4)(d), and agree with  $y_j(x,t)$ ,  $z_j(x,t)$ , respectively, for  $(x,t) \in R_j$ .

Furthermore, suppose that there is a function  $W(x,t)$  defined in  $R$  such that

- (5.2) (a)  $W$  is continuous in  $R$  and  $C^{(1)}$  in each  $R_j$ ,  $j = 1, \dots, n$ ;
- (b)  $W(x,T) = 0$ ;
- (c) for  $(x,t)$  in  $R_j$ ,  $W$  satisfies
- $$d_x(x,t) = \lambda_j(x,t),$$
- $$W_t(x,t) = -\psi_j(x,t) = -\{f(x, y_j, z_j) + \lambda_j g(x, y_j, z_j)\}.$$

A region  $R$  satisfying (5.1) and a function  $W$  satisfying (5.2) are said to constitute a field  $P$ .

The justification for the notion of a field is to be found in the following discussion. Associated with a field  $P$  are

functions  $y^*$  and  $z^*$ . There exist nonempty families of functions  $\mathcal{Y}$  and  $\mathcal{Z}$  which satisfy (2.4) and contain  $y^*$  and  $z^*$ , respectively. For example,  $\mathcal{Y}$  can be taken to be the set of all  $y(x,t)$  which are  $C^{(1)}$  in  $\bar{R}_j$  and coincide with  $y^*(x,t)$  in a neighborhood of the boundary of  $\bar{R}_j$ , for each  $j = 1, \dots, n$ . The class  $\mathcal{Z}$  can be defined similarly. We emphasize that this particular pair is offered only as an example; in what follows  $\mathcal{Y}$  and  $\mathcal{Z}$  are arbitrary, subject only to (2.4) and the condition  $y^* \in \mathcal{Y}$ ,  $z^* \in \mathcal{Z}$ .

Theorem 2. Let  $F$  be a field and let the notation be as in (5.1) and (5.2). Then, for any  $(x_0, t_0)$  in  $\bar{R}$ , the pair of functions  $y^*(x,t)$ ,  $z^*(x,t)$  is a saddle-point relative to  $\mathcal{Y}$ ,  $\mathcal{Z}$  for the initial conditions  $(x_0, t_0)$ , and  $W(x_0, t_0)$  is the value of the game.

Proof. Any solution  $x_0(t)$  of

$$\frac{dx}{dt} = g[x, y^*(x,t), z^*(x,t)] ,$$

with associated  $y_0(t) = y^*(x_0(t), t)$ ,  $z_0(t) = z^*(x_0(t), t)$ , is an extremal, by (5.1). Along the extremal,

$$\frac{dW(x_0(t), t)}{dt} = W_x \dot{x}_0 + W_t = 1g - (f + 1g) = -f(x_0(t), y_0(t), z_0(t)),$$

except for a finite exceptional set  $E_0$  of values of  $t$ . At the exceptional points, however, both right- and left-hand derivatives exist. Since  $W$  is continuous we may write

$$W(x_0, t_0) = \int_{t_0}^T \left( \frac{dW}{dt} \right) dt = \int_{t_0}^T f(x_0(t), y_0(t), z_0(t)) dt,$$

and so

$$W(x_0, t_0) = P_{(\kappa)}(y^*, z^*; x_0, t_0)$$

for any  $\kappa$  and  $(x_0, t_0)$  in  $\bar{R}$ . Let  $y$  be an element of  $\mathcal{Y}$ , and let  $x_1(t)$ ,  $y_1(t)$ ,  $z_1(t)$  denote any path corresponding to the strategies  $y(x, t)$ ,  $z^*(x, t)$ . Along this path

$$\frac{dW}{dt} = W_x \dot{x}_1 + W_t = W_x g(x_1, y, z^*) + W_t$$

except for a finite number of exceptional points which constitute a set  $E_1$ . Using (4.6), we obtain for all  $t$  not lying in  $E_1$

$$\frac{dW}{dt} \leq -f(x_1, y_1, z_1).$$

At the exceptional points both right- and left-hand limits of  $dW/dt$  as well as  $f(x_1, y_1, z_1)$  exist, so properly interpreted the inequality holds along the entire path. It follows that

$$W(x_0, t_0) \geq \int_{t_0}^T f(x_1, y_1, z_1) dt,$$

and so

$$W(x_0, t_0) \geq P_{(\mu)}(y, z^*; (x_0, t_0))$$

for all indices  $\mu$  and functions  $y$  in  $\mathcal{Y}$ . A similar argument can be used to show that

$$W(x_0, t_0) \leq P_{(\lambda)}(y^*, z; (x_0, t_0))$$

for any  $z$  in  $\mathcal{Z}$  and index  $\lambda$ . The theorem is thus proved.

## 6. SUFFICIENT CONDITIONS FOR A FIELD

In this section we proceed to find conditions under which a family of extremals defines a field. The most essential feature which the family should possess is that, with the exception of certain boundary points, each  $(x_0, t_0)$  of the region covered has a uniquely determined extremal of the family passing through it.

We may then define  $y^*, z^*$ , and  $\Lambda$  at  $(x_0, t_0)$  to agree with the  $y, z$ , and  $\lambda$ , respectively, associated with the extremal. A precise formulation of this intuitive idea is given in Theorem 3. Since it is not possible in general to obtain the solution with a single family of extremals, the idea expressed above is generalized to include  $m$  families of extremals, each simply covering a region  $R_i$ ,  $i = 1, \dots, m$ .

Theorem 3. Let  $R$  be a region with piecewise smooth boundary, and let there be a decomposition of  $R$  into nonoverlapping subregions  $R_i$ ,  $i = 1, \dots, m$ , such that  $R = \bar{R}_1 \vee \dots \vee \bar{R}_m$ . For each  $i$ ,  $i = 1, \dots, m$ :

1. Let  $U_i$  denote the closed region of the  $(t, u)$  plane bounded by the curves  $u = u_1$ ,  $u = u_2$  ( $+\infty > u_2 > u_1 > -\infty$ ),  $t = T$ , and  $t = t_1(u)$ , where  $t_1(u)$  is defined and is piecewise  $C^{(2)}$  without cusps at corners on the interval  $u_1 \leq u \leq u_2$ , and where  $t_1(u) \leq T$ , with equality permitted only for  $u = u_2$  or  $u = u_1$ . Furthermore, let there exist a family of curves  $t_{1\kappa}(u)$ ,  $\kappa = 0, 1, \dots, \kappa(1)$ , defined

and piecewise  $c^{(2)}$  without cusps at corners  
on the interval  $u_1 \leq u \leq u_f$ , such that

$$t_1(u) = t_{10}(u) \leq t_{11}(u) \leq \dots \leq t_{1k(1)}(u) = T$$

for all  $u_1 \leq u \leq u_f$ , with equalities permitted  
only for  $u = u_f$  or  $u = u_1$ . Denote by  $U_{1k}$  the  
nonoverlapping subregions defined by the curves  
 $t_{1k}(u)$ , the index  $k$  of the region being in agree-  
ment with the index of the right-hand boundary  
curve.

2. Let there exist a one-parameter family of extremals  
 $x_1(t,u)$ ,  $y_1(t,u)$ ,  $z_1(t,u)$  with associated functions  
 $\lambda_1(t,u)$  such that:

- (a)  $x_1(t,u)$  is continuous on  $U_1$  and maps  $U_1$   
in a one-to-one fashion onto  $\bar{R}_1$ .
- (b) For each  $k$ ,  $x_1(t,u)$  coincides with a function  
 $c^{(2)}$  in  $U_{1k}$ ;  $y_1(t,u)$ ,  $z_1(t,u)$  coincide with  
functions  $c^{(1)}$  in  $U_{1k}$ .
- (c)  $|x_{1u}(t,u)| \geq d > 0$  on  $U_1$ , where  $d$  is an  
appropriate constant.
- (d) The functions  
$$H_1(t,u) = f_y(x_1, y_1, z_1) + \lambda_1(t,u)g_y(x_1, y_1, z_1),$$
$$K_1(t,u) = f_z(x_1, y_1, z_1) + \lambda_1(t,u)g_z(x_1, y_1, z_1),$$
where  $x_1 = x_1(t,u)$ ,  $y_1 = y_1(t,u)$ ,  $z_1 = z_1(t,u)$ ,  
are both continuous on  $U_1$ ; moreover,  $H_1$  vanishes  
at any point where  $y_1$  is discontinuous, and  $K_1$   
vanishes at any point where  $z_1$  is discontinuous.

3. Let the functions  $w_1(t, u)$ , defined by

$$(6.1) \quad w_1(t, u) = \int_t^T f[x_1(t, u), y_1(t, u), z_1(t, u)] dt, \quad i=1, \dots, m,$$

be such that whenever  $x_1(t, u) = x_j(t, u')$  then

$$w_1(t, u) = w_j(t, u').$$

Then the  $m$  families  $x_1(t, u), \dots, x_m(t, u)$  determine a field  $P$  over  $R$ .

Proof. It is required to show that (5.1) and (5.2) hold.

From hypotheses 1 and 2 it can easily be shown that corresponding to each  $U_{1k}$  there is determined by means of  $x_1(t, u)$  a simply connected region  $R_{1k}$  with piecewise smooth boundary and such that for,  $i, k \neq j, \ell$ ,  $R_{1k} \cap R_{j\ell} = \emptyset$ . These regions  $R_{1k}$ , suitably renumbered, can be taken as the subregions  $R_j$  of  $R$  described in (5.1)(a). It also follows from the hypotheses of the theorem that for each  $i$  the functions  $x_i(t, u)$  can be inverted to give functions  $u = u_i(x, t)$ , continuous on  $\bar{R}_i$  and  $C^{(2)}$  in each  $R_{1k}$ . For  $(x, t)$  in  $R_{1k}$  define:

$$(6.2) \quad \begin{aligned} y_{1k}^*(x, t) &= y_1(t, u_1(x, t)), \\ z_{1k}^*(x, t) &= z_1(t, u_1(x, t)), \\ \lambda_{1k}(x, t) &= \lambda_1(t, u_1(x, t)). \end{aligned}$$

These functions clearly satisfy (4.1)(a) and (4.1)(b) on  $R_{1k}$ .

Since  $x_1(t, u), y_1(t, u), z_1(t, u)$  is an extremal, and, by (2.1)(b),  $f + \lambda_{1k}g$  is concave in  $y$  and convex in  $z$ , it follows that (4.1)(c) also holds on  $R_{1k}$ . From the definition (3.3) of  $\lambda_1$  and the relation

$$\begin{aligned} \lambda_{1t} &= \lambda_{1kx} x_{1t} + \lambda_{1kt} = \lambda_{1kx} g(x_1, y_1, z_1) + \lambda_{1kt} \\ &= \lambda_{1kx} g(x_1, y_{1k}^*, z_{1k}^*) + \lambda_{1kt}, \end{aligned}$$

it follows that (4.1)(d) holds. Hence (5.1)(a) is verified.

To show that (5.1)(b) holds, we proceed as follows. Define

$$y^*(x,t) = y_{1k}^*(x,t)$$

$$z^*(x,t) = z_{1k}^*(x,t) \quad \text{for } i = 1, \dots, m; k = 1, \dots, k(i),$$

if  $(x,t)$  belongs to just one region  $R_{1k}$ . If  $(x,t)$  belongs to more than one region  $R_{1k}$  then we choose the first such region according to the lexicographic ordering of  $(i,k)$  to define  $y^*$  and  $z^*$ . With this definition,  $y^*$  and  $z^*$  satisfy (2.4)(a). For the purpose of checking that (2.4)(c) and (d) hold, it suffices to show that for any  $i,k,u$ , the curve  $x_1(t,u)$  is not tangent to the curve  $X(u) = x_1(t_{1k}(u),u)$ . This is immediate, for tangency at a point would imply the existence of a constant  $\beta$  such that

$$dX = x_{1t} dt_{1k} + x_{1u} du = \beta x_{1t} dt,$$

$$dt_{1k} = \beta dt,$$

whence

$$x_{1u} du = 0.$$

But  $du \neq 0$  since  $t_{1k}(u)$  is  $C^{(2)}$ , and  $x_{1u}$  is bounded away from zero; thus the curves cannot be tangent.

We now proceed to verify (5.2). Define

$$(6.3) \quad w(x,t) = w_1(t, u_1(x,t)) \quad \text{for } (x,t) \in R_1.$$

From hypothesis 3 it is clear that  $w(x,t)$  is continuous on  $R$  and is  $C^{(1)}$  on each  $R_{1k}$ . It is evident from (6.1) and (6.3) that  $w(x,T) = 0$ . Thus (5.2)(a) and (5.2)(b) are established.

On each  $R_{1k}$  we have, by (6.3),

$$w(x(t,u),t) = w_1(t,u), \quad u = u_1(x,t),$$



whence

$$w_x x_t + w_t = w_{1t}.$$

But, by (6.1),  $w_{1t} = -f$  and  $x_t = g$ , so we have

$$w_t(x, t) = -f(x, y^*, z^*) - w_x(x, t)g(x, y^*, z^*).$$

Thus to establish (5.2)(c) it is required to show that  $w_x = \hat{\lambda}_{1k}$  on each  $R_{1k}$ . That is to say (see (6.2)), it suffices to show that

$$(6.4) \quad \hat{\lambda}_1(t, u) = \hat{\lambda}_1(t, u_1(x, t)) = w_{1u} u_{1x}.$$

Differentiating (6.1) yields

$$(6.5) \quad w_{1u} = \int_t^T \frac{\partial f}{\partial u} dt + \sum_{t_{1k} > t} \Delta_{1k}(f) \frac{dt_{1k}}{du},$$

where

$$\begin{aligned} \frac{\partial f}{\partial u} &= f_x x_{1u} + f_y y_{1u} + f_z z_{1u}, \\ \Delta_{1k}(f) &= f[x(t_{1k}, u), y^-(t_{1k}, u), z^-(t_{1k}, u)] \\ &\quad - f[x(t_{1k}, u), y^+(t_{1k}, u), z^+(t_{1k}, u)], \end{aligned}$$

the + and - superscripts indicating right- and left-hand limits as  $(t, u) \rightarrow (t_{1k}(u), u)$  from the interior of  $U_{1,k+1}$  and  $U_{1,k}$ , respectively. Upon differentiating the right-hand side of (6.4) with respect to  $t$ , we get, with the help of (6.5),

$$(w_{1u} u_{1x})_t = - \frac{\partial f}{\partial u} u_{1x} + w_{1u} u_{1xt}, \quad (x, t) \text{ in some } R_{1k}.$$

But  $u_{1x} = (x_{1u})^{-1}$ , and so

$$\frac{\partial f}{\partial u} u_{1x} = f_x + f_y y_{1u} + f_z z_{1u} u_{1x}.$$

Hence, by (6.2),

$$(6.6) \quad \frac{\partial f}{\partial u} u_{1x} = f_x + f_y \frac{\partial y_{1k}^*}{\partial x} + f_z \frac{\partial z_{1k}^*}{\partial x} = \frac{\partial f}{\partial x}.$$

Furthermore,

$$w_{1u} u_{1xt} = - \frac{w_{1u} x_{1ut}}{(x_{1u})^2} = - \left( \frac{w_{1u}}{x_{1u}} \right) \left( \frac{\partial g}{\partial u} \right) \left( \frac{1}{x_{1u}} \right),$$

where

$$\frac{\partial g}{\partial u} = g_x x_{1u} + g_y y_{1u} + g_z z_{1u}.$$

Thus, writing

$$(6.7) \quad \frac{\partial g}{\partial x} = g_x + g_y \frac{\partial y_{1k}^*}{\partial x} + g_z \frac{\partial z_{1k}^*}{\partial x},$$

we have

$$(w_{1u} u_{1x})_t = \frac{\partial f}{\partial x} - (w_{1u} u_{1x}) \frac{\partial g}{\partial x}.$$

By using (4.2) one can write the differential equation (3.3)

for  $\lambda$  as follows:

$$\lambda_t = - \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x},$$

where  $f/x$  and  $g/x$  are as defined in (6.6) and (6.7). Thus for each  $u_1 \leq u \leq u_1^*$ ,  $\lambda_1(t, u)$  and  $w_{1u} u_{1x}$  satisfy the same differential equation in  $t$  at all interior points of the regions  $U_{1k}$ . Furthermore, since  $w_{1u}(T, u) = 0$ ,  $\lambda_1(T, u) = 0$ , it follows that to complete the proof of (6.4) it is necessary to show that  $w_{1u} u_{1x}$  is continuous across the curves  $t_{1k}(u)$ . If we drop the subscript 1 and use the superscripts + and - as explained above, then for each  $k$ ,  $k = 1, \dots, k(1)-1$ , it is required to show that

$$(w_u u_x)^- - (w_u u_x)^+ = 0.$$

Now

$$\begin{aligned}
 (w_u u_x)^- - (w_u u_x)^+ &= (w_u^- - w_u^+) u_x^- + w_u^+ (u_x^- - u_x^+) \\
 (6.8) \qquad &= \frac{1}{x_u^-} (w_u^- - w_u^+) + w_u^+ \left( \frac{1}{x_u^-} - \frac{1}{x_u^+} \right) \\
 &= \frac{1}{x_u^-} \left\{ (w_u^- - w_u^+) + w_u^+ u_x^+ (x_u^+ - x_u^-) \right\} .
 \end{aligned}$$

Clearly, (6.4) is valid on  $U_{1k}(1)$ ; so, proceeding inductively, we may assume that  $w_u^+ u_x^+ = \lambda$ . It follows from (6.5) that

$$w_u^+ - w_u^- = -[f(x, y_1^+, z_1^+) - f(x, y_1^-, z_1^-)] \frac{dt_k}{du} .$$

Writing  $x$  as the integral of  $g$ , and then differentiating with respect to  $u$ , gives

$$x_u^+ - x_u^- = -[g(x, y_1^+, z_1^+) - g(x, y_1^-, z_1^-)] \frac{dt_k}{du} .$$

Thus the right-hand member of (6.8) can be written as

$$u_x^- \left( \frac{dt_k}{du} \right) \left\{ [f(x, y_1^+, z_1^+) - f(x, y_1^-, z_1^-)] - [g(x, y_1^+, z_1^+) - g(x, y_1^-, z_1^-)] \right\} ,$$

and to complete the proof we must show that the term in curly braces is zero.

If  $y^+ = y^-$ ,  $z^+ = z^-$ , then there is no problem. Suppose  $y^- \neq y^+$  but  $z^- = z^+ = z$ . Then by hypothesis 2(d),  $H = f_y + \lambda g_y = 0$ . It follows from the concavity of

$$\phi(y, z) = f(x, y, z) + \lambda g(x, y, z)$$

in  $y$  that both  $y^+$  and  $y^-$  maximize  $\phi$  for the given  $z$  and  $x$  in question. Hence

$$f(x, y^+, z) - f(x, y^-, z) = -\lambda [g(x, y^+, z) - g(x, y^-, z)] ,$$

which, since  $z = z^- = z^+$ , says that the term in curly braces is zero. Suppose then that  $y^- \neq y^+$ ,  $z^- \neq z^+$ . Then  $H = K = 0$ , and  $(y^-, z^-)$  and  $(y^+, z^+)$  are both saddle-points for the game over the square with payoff  $\Phi(y, z)$ . Hence  $\Phi(y^-, z^-) = \Phi(y^+, z^+)$ , and

$$f(x, y^+, z^+) - f(x, y^-, z^-) = \lambda [g(x, y^+, z^+) - g(x, y^-, z^-)] ,$$

as was to be proved.

## 7. AN EXAMPLE

We shall consider the game for which

$$f(x,y,z) = x^2, \quad g(x,y,z) = 2y - \frac{z}{2} - 1, \quad x(0) = c.$$

Let us make some preliminary observations. Intuitively, it is desirable for Player I to make  $x$  either very large or very small if possible. Further, given sufficient time he can reach either of these goals, as is shown by the following table which gives the values of  $dx/dt$  for the extreme values of  $y$  and  $z$ .

	$z = 0$	$z = 1$
$y = 0$	-1	$-3/2$
$y = 1$	+1	$+1/2$

This suggests that a good choice for I may be either

$$y_0(x,t) = y_0(t) = 0 \quad \text{or} \quad y_1(x,t) = y_1(t) = 1.$$

Moreover, if  $c > 0$  and I chooses  $y_1$ , II should clearly choose  $z_1 = 1$ . Similarly, if  $c < 0$  and I chooses  $y_0$ , II should choose  $z_0 = 0$ . We have

$$\begin{aligned}
 P(y_0, z_0) &= \int_0^T (c-t)^2 dt = c^2 T - cT^2 + \frac{T^3}{3}, \\
 (7.1) \quad P(y_0, z_1) &= \int_0^T (c - \frac{3}{2}t)^2 dt = c^2 T - \frac{3}{2} cT^2 + \frac{3}{4} T^3, \\
 P(y_1, z_0) &= \int_0^T (c+t)^2 dt = c^2 T + cT^2 + \frac{T^3}{3}, \\
 P(y_1, z_1) &= \int_0^T (c + \frac{1}{2}t)^2 dt = c^2 T + \frac{1}{2} cT^2 + \frac{T^3}{12}.
 \end{aligned}$$

Clearly,

$$\begin{aligned} P(y_0, z_0) &> P(y_1, z_1) && \text{if } T > 6u, \\ &< P(y_1, z_1) && \text{if } T < 6u. \end{aligned}$$

Next, by (3.2) and (3.3),

$$\begin{aligned} H &= f_y + \lambda g_y = 2\lambda, \\ K &= f_z + \lambda g_z = -\frac{1}{2}\lambda, \\ \frac{d\lambda}{dt} &= -(f_x + \lambda g_x) = -2x, && \lambda(T) = 0. \end{aligned}$$

Set  $x(T) = u$ . For  $u > 0$ ,  $d\lambda/dt < 0$  at  $t = T$ , and hence  $\lambda > 0$  in some interval  $t_1 \leq t \leq T$ . Then  $H > 0$ ,  $K < 0$  on  $(t_1, T)$ , and so  $y(t, u) = 1$ ,  $z(t, u) = 1$  and  $dx(t, u)/dt = 1/2$  along the extremal passing through  $(u, T)$ . Along this extremal,  $x(t, u) = (t-T)/2 + u$ . The value of  $t_1$  is determined by  $\lambda(t_1) = 0$  to be  $t_1 - T = -4u$ . At  $t = t_1$ , we have  $x(t_1, u) = -u$ . Similarly for  $u < 0$ , we have  $y(t, u) = 0$ ,  $z(t, u) = 0$  on an interval  $t_2 \leq t \leq T$ ; along this extremal, we have  $x(t, u) = -(t - T) + u$ , the value of  $t_2$  is  $t_2 - T = 2u$ , and  $x(t_2, u) = -u$ . We have thus defined two one-parameter families of extremals, one family corresponding to values  $u > 0$ , and the other to values  $u < 0$ . Consider the first family on the closed region  $U_1^*$  defined by  $u \geq 0$ ,  $T - (3u/2) \leq t \leq T$ , and the second family on the region  $U_2^*$  defined by  $u \leq 0$ ,  $T + (6u/5) \leq t \leq T$ . (See Fig. 2.) The function  $x(t, u) = (t-T)/2 + u$  maps the interior of  $U_1^*$  in a one-to-one fashion onto the region  $R_1^*$  of the  $(x, t)$  plane defined by  $t < T$  and  $x > -(t-T)/6$ . (See Fig. 2b.) On the other hand, the function  $x(t, u) = -(t-T)+u$  maps the interior of  $U_2^*$  onto the region  $R_2^*$  which is defined by

$t < T$  and  $x < -(t-T)/6$ . Denoting functions from the first family by the subscript 1 and those from the second by 2, we have, for arbitrary  $t_0 \leq T$ ,

$$x_1(t_0, -s/s(t_0-T)) = x_2(t_0, s/s(t_0-T))$$

and

$$\int_{t_0}^T \left\{ \frac{(t-T)}{2} - s/s(t_0-T) \right\}^2 dt = \int_{t_0}^T \left\{ (t-T) + s/s(t_0-T) \right\}^2 dt;$$

that is,

$$w_1(t_0, -s/s(t_0-T)) = w_2(t_0, s/s(t_0-T)).$$

Thus we have a field over the half plane  $t \leq T$  with two subregions  $R_1^*$  and  $R_2^*$ . To place the example completely in the context of the theory we restrict  $U_1^*$  and  $U_2^*$  to nonnegative  $t$  values and bounded  $u$  values in order to obtain regions  $U_1$  and  $U_2$ , respectively, and thereby obtain in turn regions  $R_1$  and  $R_2$  which are restrictions of  $R_1^*$  and  $R_2^*$ , respectively, and lie in the strip  $0 < t < T$  of the  $(x, t)$  plane. A saddle-point is the pair of functions:

$$y^*(x, t) = \begin{cases} 1 & \text{if } (x, t) \text{ is in } R_1 \text{ or in } \bar{R}_1 \wedge \bar{R}_2, \\ 0 & \text{if } (x, t) \text{ is in } R_2; \end{cases}$$

$$z^*(x, t) = \begin{cases} 1 & \text{if } (x, t) \text{ is in } R_1 \text{ or in } \bar{R}_1 \wedge \bar{R}_2, \\ 0 & \text{if } (x, t) \text{ is in } R_2. \end{cases}$$

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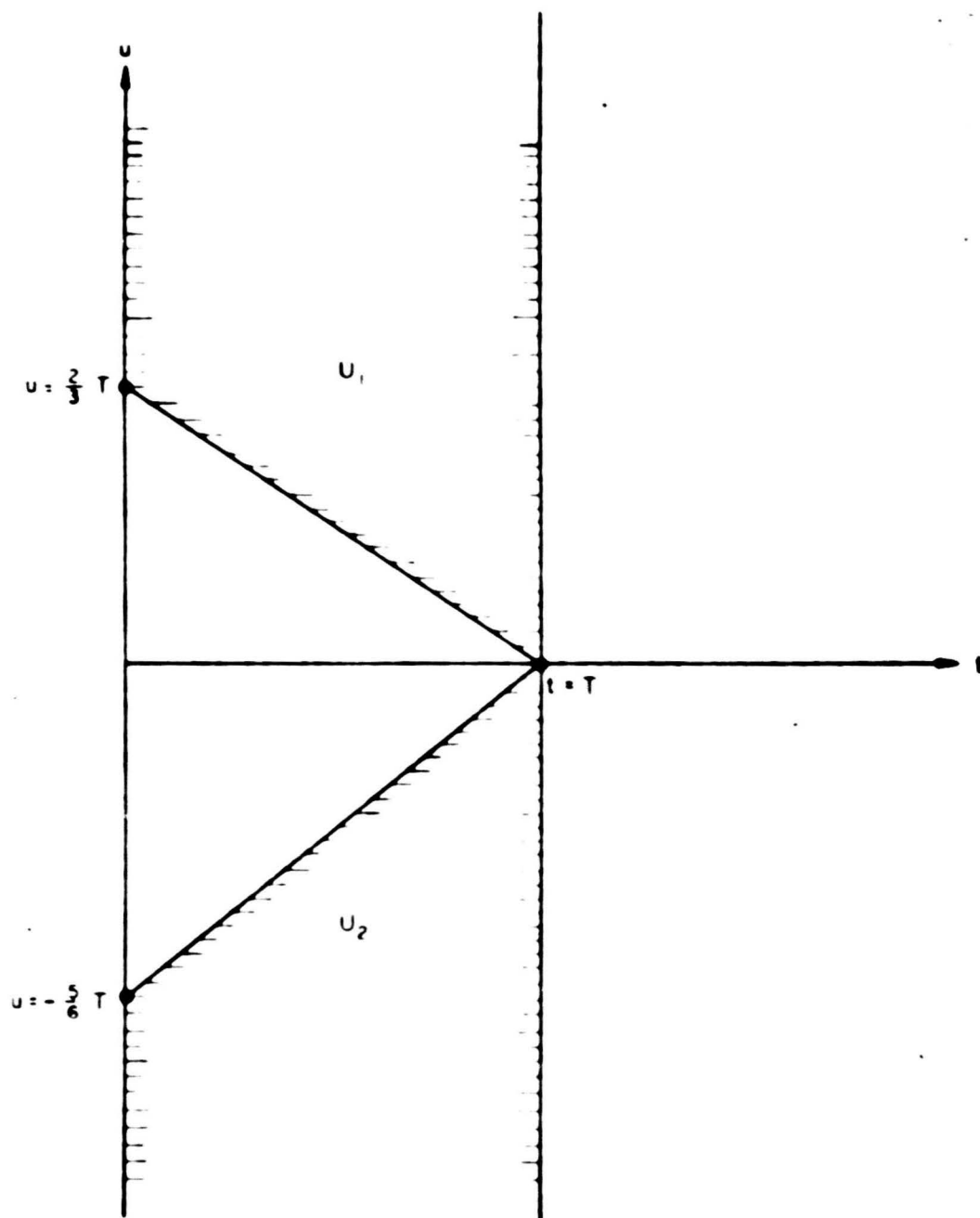


Fig. 2a



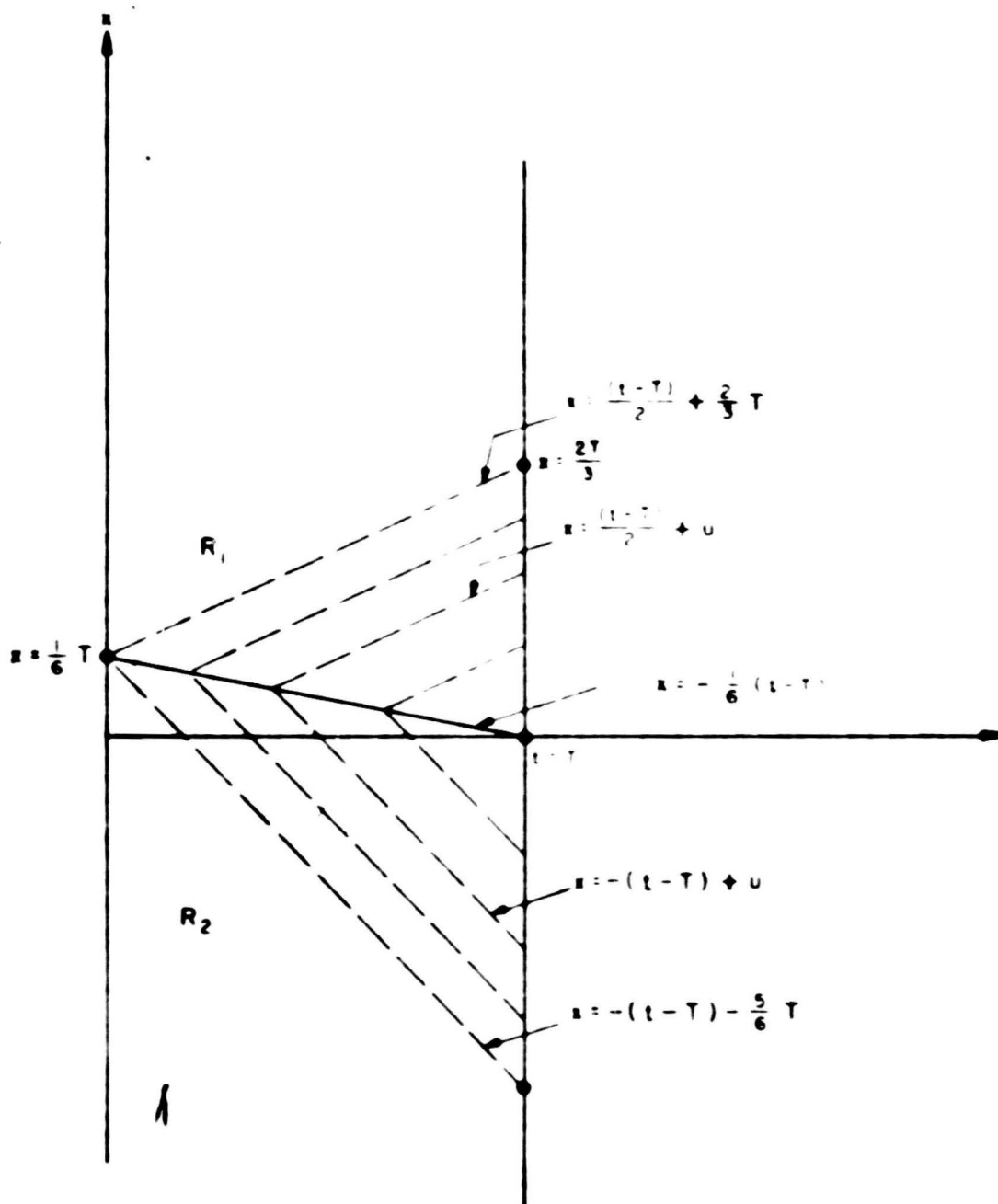


Fig. 2b

Dashed lines are extremals

REFERENCES

- [1] Bellman, R., Dynamic Programming of Continuous Processes, The RAND Corporation, Report R-271, July, 1954.
- [2] Bellman, R., W. H. Fleming, and D. V. Widder, "Variational Problems with Constraints," Annali di Matematica, Vol. 41 (to appear).
- [3] Bliss, G. A., Lectures on the Calculus of Variations, The University of Chicago Press, Chicago, 1946.
- [4] Fleming, W. H., Relative Maxima in Variational Problems, with Inequality Constraints (unpublished).
- [5] ———, "A Note on Differential Games of Prescribed Duration," Contributions to The Theory of Games, III, Princeton (to appear).
- [6] Isaacs, R. P., Differential Games II: The Definition and Formulation (to appear).
- [7] ———, Differential Games III: The Basic Principle of the Solution Process (to appear).
- [8] Valentine, F. A., "The Problem of Lagrange with Differential Inequalities as Added Side Conditions," Contributions to the Calculus of Variations, 1933-1937, University of Chicago Press, Chicago, 1937.